Math 237Y- 2016-2017

Term Test 5 - March 24, 2017

Time allotted: 110 minutes. Aids permitted: None.

Total marks: 70

Full Name: ______________________________
Last First

Student Number: ______________________

Utoronto Email: ______________________ **@mail.utoronto.ca**

Instructions

- DO NOT WRITE ON THE QR CODE at the top of the pages.
- DO NOT DETACH ANY PAGE.
- NO CALCULATORS or other aids allowed.
- Unless otherwise stated, you must JUSTIFY your work to receive credit.
- Check to make sure your test has all 10 pages.
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- DO NOT START the test until instructed to do so.

GOOD LUCK!

1. Find the arclength of:

(a) (5 points) The curve given by $(x, y) = (\cos^3(t), \sin^3(t))$ for $0 \leqslant t \leqslant \frac{\pi}{2}$

Solution

Let $\gamma(t) = (\cos^3(t), \sin^3(t))$. Then $\gamma'(t) = 3\sin(t)\cos(t)(-\cos(t), \sin(t))$ and $|\gamma'(t)| = 3|\sin(t)\cos(t)|$. The arclength is given by

$$\int_0^{\pi/2} |\gamma'(t)|\, dt = 3\int_0^{\pi/2} |\sin(t)\cos(t)|\, dt = 3\int_0^{\pi/2} \sin(t)\cos(t)\, dt = \frac{3}{2}.$$

(b) (5 points) The graph of a C^1 function $y = f(x)$ for $a \leqslant x \leqslant b$

Solution Parametrize the graph of f by $\gamma(t) = (t, f(t))$. Then the arclength is given by

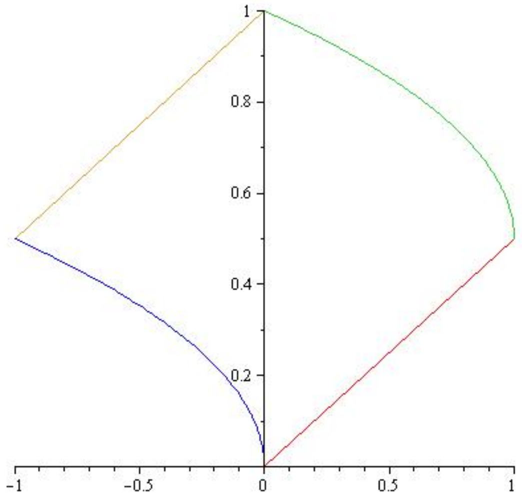
$$\int_a^b |\gamma'(t)|\, dt = \int_a^b |(1, f'(t))|\, dt = \int_a^b \sqrt{1 + (f'(t))^2}\, dt$$

2. Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(u, v) = (x, y) = (u - v^2, \frac{1}{2}(u + v))$. Let B be the square with endpoints $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$ in the (u, v) plane. Let $A = T(B)$.

(a) (5 points) Draw a picture of the set A and label the equations of all the boundary curves.

Solution

The set A is as follows.



The vertices, starting from the origin and going counterclockwise, are $(0, 0), (1, 1/2), (0, 1), (-1, 1/2)$. The boundary curves starting from the bottom right and going counterclockwise are: $(u, u/2)$ for u from 0 to 1, $(1 - v^2, (1 + v)/2)$ for v from 0 to 1, $(u - 1, (u + 1)/2)$ for u from 1 to 0, and $(-v^2, v/2)$ for v from 1 to 0.

(b) (5 points) Calculate the area of the set A.

Solution

By change of variables, the area is given by

$$\iint_A dA = \iint_B |\det DT|\, dA = \iint_B \left|\det \begin{pmatrix} 1 & -2v \\ 1/2 & 1/2 \end{pmatrix}\right| dA = \int_0^1 \int_0^1 |1/2 + v|\, du dv = 1.$$

3. Let $\mathbf{F}(x, y, z) = (x^2, x^2y, z + zx)$

(a) (5 points) Verify that $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.

Solution

One computes

$$\nabla \times \mathbf{F} = \left(\frac{\partial}{\partial y}(z + zx) - \frac{\partial}{\partial z}x^2y, \frac{\partial}{\partial z}x^2 - \frac{\partial}{\partial x}(z + zx), \frac{\partial}{\partial x}(x^2y) - \frac{\partial}{\partial y}x^2\right) = (0, -z, 2xy)$$

and thus

$$\nabla \cdot (\nabla \times \mathbf{F}) = \frac{\partial}{\partial x}0 - \frac{\partial}{\partial y}z + \frac{\partial}{\partial z}2xy = 0.$$

(b) (5 points) Can there exist a function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$? Justify.

Solution

Suppose yes. That is, suppose $\mathbf{F} = \nabla f$. Then $\nabla \times \mathbf{F} = \mathbf{0}$, since $\nabla \times \nabla f = \mathbf{0}$, for all $f \in C^2$. But from part of our computation in part (a) we see that $\nabla \times \mathbf{F} \neq \mathbf{0}$. A contradiction. Thus, **the answer is no**.

4. (a) (5 points) Let $\gamma(t) = (1, t, t^2)$, $0 \leqslant t \leqslant 5$ be a curve, and consider the vector field $\mathbf{F}(x, y, z) = (1, 2y, -1)$. Evaluate $\int_\gamma \mathbf{F} \cdot d\mathbf{x}$.

   Solution

   Since $\gamma'(t) \cdot \mathbf{F}(\gamma(t)) = 0$, we see that $\int_\gamma \mathbf{F} \cdot d\mathbf{x} = \int_0^5 0 dt = 0$

   (b) (5 points) Let $\mathbf{F}(x, y, z) = (x^2yz, xy^2z, xyz^2)$, and let γ be the oriented curve (say oriented counterclockwise) that is obtained by intersecting the sphere $x^2 + y^2 + z^2 = 1$ with the cone $x^2 + y^2 = z^2$, $z > 0$. Determine $\int_\gamma \mathbf{F} \cdot d\mathbf{x}$.

   Solution

   Since $\mathbf{F}$ is normal to the sphere, it is normal to any curve on the sphere. Alternatively we compute that $\gamma = \{(x, y, z) : x^2 + y^2 = \frac{1}{2}, z = \frac{1}{\sqrt{2}}\} = \{(\frac{1}{\sqrt{2}} \cos t, \frac{1}{\sqrt{2}} \sin t, \frac{1}{\sqrt{2}})\}$ and check that $\mathbf{F}(\gamma(t)) \cdot \gamma'(t) = 0$. Thus $\int_\gamma \mathbf{F} \cdot d\mathbf{x} = 0$.

5. (10 points) Find the flux of $\mathbf{F}(x, y, z) = (x^2, y, 1)$ through the surface $S = \{(x, y, z) : -2x + y + z = 0, (x, y) \in [0, 2] \times [0, 2]\}$, oriented with the "downwards" orientation.

(**Note:** The "downwards" orientation means that the surface is oriented with the normal vector $\mathbf{n}$ to the surface that has negative z-component)

Solution

Parametrize S by $p(x, y) = (x, y, 2x - y)$, $(x, y) \in [0, 2] \times [0, 2]$. Denote $N = \frac{\partial p}{\partial x} \times \frac{\partial p}{\partial y} = (1, 0, 2) \times (0, 1, -1) = (-2, 1, 1)$, which is the pointed in the opposite direction to our orientation. Then the flux is $\Phi = -\int_S \mathbf{F} \cdot \frac{N}{\|N\|} dA = -\int_{[0,2]^2} \mathbf{F} \cdot N dx dy$. That is

$$\Phi = \int_0^2 \int_0^2 (2x^2 - y - 1) dx dy = 4 \int_0^2 x^2 dx - 2 \int_0^2 y dy - 4 = \frac{8}{3}$$

6. Let $\mathbf{F} = (\frac{2x}{x^2 + e^y}, \frac{e^y}{x^2 + e^y})$ be a vector field in $\mathbb{R}^2$.

(a) (5 points) Let S be a triangle in $\mathbb{R}^2$. Show that $\mathbf{F}$ is exact in S without computing integrals.

Solution

Check that $\mathbf{F}$ is C^1, $\partial_x \frac{e^y}{x^2+e^2} = \partial_y \frac{2x}{x^2+e^y} \iff -2xe^y = -2xe^y$, and note that S is star-shaped.

(b) (5 points) Find a scalar potential for $\mathbf{F}$ in $\mathbb{R}^2$ (you may compute integrals).

Solution

Integrating the first component of $\mathbf{F}$ with respect to x gives $\log(x^2 + e^y)$ plus a function only dending on y, and similarly integrating the second component of $\mathbf{F}$ with respect to y gives $\log(x^2 + e^y)$ plus a function only depending on x. Hence one can take $f(x, y) = \log(x^2 + e^y)$.

7. (a) (5 points) Let γ be the boundary of the rectangle with vertices $(\pi,-1)$, $(-\pi,-1)$, $(-\pi,1)$, $(\pi,1)$, oriented clockwise. Use Green's Theorem to evaluate $\displaystyle\int_\gamma (e^y\,dx + \sin x\,dy)$

Solution

By Green's theorem,

$$\int_\gamma (e^y dx + \sin x dy) = -\int_{[-\pi,\pi]\times[-1,1]} (\cos x - e^y) dx dy = 2\pi(e - 1/e)$$

(b) (5 points) Let $\gamma : [a,b] \to \mathbb{R}^2$ be a C^1 curve in $\mathbb{R}^2$ such that for any C^1 function $f : \mathbb{R}^2 \to \mathbb{R}$ it holds that $\int_\gamma \nabla f \cdot d\mathbf{x} = 0$. Prove that γ is a closed curve.

Solution

Recall that $\int_\gamma \nabla f(x,y) \cdot d\mathbf{x} = f(\gamma(b)) - f(\gamma(a))$. If $\gamma(a) \neq \gamma(b)$ we could take $f(\mathbf{p}) = \|\mathbf{p} - \gamma(a)\|^2$ and get a contradiction, since $f(\gamma(b)) \neq f(\gamma(a))$.

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