Math 237Y- 2016-2017

Term Test 4 - February 17, 2017

Time allotted: 110 minutes. Aids permitted: None.

Total marks: 70

Full Name: ____________________
Last First

Student Number: ____________________

Utoronto Email: ____________________ **@mail.utoronto.ca**

Instructions

- DO NOT WRITE ON THE QR CODE at the top of the pages.
- DO NOT DETACH ANY PAGE.
- NO CALCULATORS or other aids allowed.
- Unless otherwise stated, you must JUSTIFY your work to receive credit.
- Check to make sure your test has all 10 pages.
- You can use the last two pages as scrap paper.
- DO NOT START the test until instructed to do so.

GOOD LUCK!

1. Determine whether the following sets S are smooth surfaces:

   (a) (5 points) $S = F^{-1}(0)$ where $F(x, y, z) = 3xy + x^2 + z$

   Solution.

   The gradient $\nabla F = (3y + 2x, 3x, 1)$ is never zero so $S = F^{-1}(0)$ is a smooth surface.

   (b) (5 points) $S = F^{-1}(0)$ where $F(x, y, z) = \cos(xy) + e^z$

   Solution.

   The gradient $\nabla F = (-y\sin(xy), -x\sin(xy), e^z)$ is never zero so $S = F^{-1}(0)$ is a smooth surface.

2. (10 points) Let $A = [0,1] \times [0,1]$. Let $f, g : A \to \mathbb{R}$ be integrable, and suppose $f \leqslant g$. **Prove, by using only a definition of the integral**, that $\int_A f \leqslant \int_A g$.

Solution.

Let $\mathcal{P}$ be a partition of A into rectangles and consider a rectangle $R \in \mathcal{P}$. Then for all $x \in R$ $f(x) \leqslant g(x)$ so $\sup_{x \in R} f(x) \leqslant \sup_{x \in R} g(x)$. Therefore

$$U(f,\mathcal{P}) = \sum_{R \in \mathcal{P}} (\sup_{x \in R} f(x)) A(R) \leqslant \sum_{R \in \mathcal{P}} (\sup_{x \in R} g(x)) A(R) = U(g,\mathcal{P})$$

and so

$$\int_A f = \inf_{\mathcal{P}} U(f,\mathcal{P}) \leqslant \inf_{\mathcal{P}} U(g,\mathcal{P}) = \int_A g.$$

3. (10 points) Let $A = [0,1] \times [0,1]$. Let $f : A \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 0 & \text{if } x \text{ is } \textbf{irr}\text{ational} \\ 0 & \text{if } x \text{ is rational, } y \text{ is } \textbf{irr}\text{ational} \\ \frac{1}{q} & \text{if } x \text{ is rational, } y = \frac{p}{q} \neq 0,\ p,q \in \mathbb{N} \text{ in lowest terms.} \\ 1 & \text{if } x \text{ is rational, } y = 0 \end{cases}$$

(Note: "in lowest terms" means that p and q have no common factor except 1)

Prove, by using only a definition of the integral, that f is integrable and that $\int_A f = 0$.

Solution.

By the density of irrationals for all rectangles $R \subseteq A$ we can find $(x,y) \in R$ with $f(x,y) = 0$ so for all partitions $\mathcal{P}$ of A, $u(f,\mathcal{P}) = 0$ and thus $\sup_{\mathcal{P}} u(f,\mathcal{P}) = 0$.

For the upper sum, let $\epsilon > 0$. We will construct a partition whose upper sum is less than ϵ. Consider a positive integer N and partition $\mathcal{P} = \{[0,1] \times [0,1/N], [0,1] \times [1/N, 2/N], \ldots, [0,1] \times [(N-1)/N, 1]\}$ of A into N equal strips. Let S_N be te set of all rational numbers $p/q \in [0,1]$ in lowest terms with $q < N$. For each $R \in \mathcal{P}$ and for each $p/q \in S_N \cap R$ subdivide R by a strip of width δ (to be determined later) around the line $y = p/q$. Call such a strip $Q_{p/q}$ and take δ small so that each $Q_{p/q}$ lie inside R and do not intersect one another, and refine $\mathcal{P}$ to a partition $\mathcal{P}'$ so that it consists of all the $Q_{p/q}$ and rectangles formed by removing from R all the $Q_{p/q}$ contained in R. By construction each $(x,y) \in R \setminus Q_{p/q}$ satisfies $f(x,y) \leqslant 1/N$. Thus the upper sum can be estimated as

$$U(f,\mathcal{P}') = \sum_{p/q \in S_N} (\sup_{Q_{p/q}} f) A(Q_{p/q}) + \sum_{R' \in \mathcal{P}' \setminus \{Q_{p/q} : p/q \in S_N\}} (\sup_{R'} f) A(R') \leqslant \#S_N \cdot \delta + \frac{1}{N}.$$

Taking $N > 2/\epsilon$ and $\delta < 1/(\#S_N \cdot N)$ gives the result.

4. (10 points) Assume that $S \subset [0,1]$ is a set with (1-dimensional) Jordan measure 0, and **prove, by using only the definition of Jordan measure** 0, that

$$T := S \times [0,1] := \{(x,y) \in \mathbb{R}^2 : x \in S,\ 0 \le y \le 1\}$$

has (2-dimensional) Jordan measure 0.

Solution.

For any $\varepsilon > 0$, we must find a finite collection of rectangles $R_1, \dots, R_n$such that

$$T \subset \cup_{j=1}^n R_j, \qquad \sum_{j=1}^n A(R_j) < \varepsilon.$$

To do this, note that because S has Jordan measure 0, there exists a finite collection of intervals, say $I_1, \dots, I_n$ for some n, such that

$$S \subset \cup_{j=1}^n I_j, \qquad \sum_{j=1}^n \text{length}(I_j) < \varepsilon.$$

For eah j, let $R_j := I_j \times [0,1]$. Then it is clear that

$$T \subset \cup_{j=1}^n R_j,$$

and since $A(R_j) = 1 \times \text{length}(I_j) = \text{length}(I_j)$ for every j, we have

$$\sum_{j=1}^n A(R_j) = \sum_{j=1}^n \text{length}(I_j) < \varepsilon.$$

5. (10 points) Evaluate the integral of the function $f(x, y) = 3x^2y^2e^{x^2y^3}$ over the set

$$\{(x, y) : 1 < x < 2, \quad y > 0, \quad xy^3 \leq 1\}$$

.

Solution.

The domain of integration may be written as the set where

$$1 \leq x \leq 2, \qquad 0 \leq y \leq x^{-1/3}.$$

Thus, the integral can be written as

$$\int_1^2 \left(\int_0^{x^{-1/3}} 3x^2y^2e^{x^2y^3}\, dy \right) dx$$

For the "inner" integral, we make the substitution $u = x^2y^3$. Then (since x is treated as a constant in this integral) $du = 3x^2y^2dy$, so the integrand is transformed into $e^u du$. Also, $y = 0$ implies that $u = 0$, and $y = x^{-1/3}$ implies that $u = x$. Thus

$$\int_0^{x^{-1/3}} 3x^2y^2e^{x^2y^3}\, dy = \int_0^x e^u du \;=\; e^x - 1.$$

So

$$\int_1^2 \left(\int_0^{x^{-1/3}} 3y^2e^{x^2y^3}\, dy \right) dx = \int_1^2 (e^x - 1)dx \;=\; (e^x - x)\Big|_1^2 = e^2 - e - 1.$$

6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continous function, and define

$$G(x) = \int_a^x (\int_{a^3}^{s^3} f(s,t)\,dt)\,ds.$$

(a) (7 points) Determine what the four limits of integration below should be.

$$G(x) = \int_{\square}^{\square} (\int_{\square}^{\square} f(s,t)\,ds)\,dt.$$

Solution.

The region of integration, call it R, is $\{(s,t) : a^3 \leqslant t \leqslant s^3,\, a \leqslant s \leqslant x\}$. Thus t ranges from a^3 to x^3 since $s \leqslant x$, and $s \geqslant t^{1/3}$. Therefore the region can also be written as $\{(s,t) : a^3 \leqslant t \leqslant x^3,\, t^{1/3} \leqslant s \leqslant x\}$.

Thus, we can write the integral with the correct limits of integration as

$$G(x) = \int_{a^3}^{x^3} \int_{t^{1/3}}^{x} f(s,t)\; ds\; dt$$

(b) (3 points) Provide a proof that your answer to part(a) is correct.

Solution.

Our answer in part(a) is correct by using Fubini's theorem. Note that the hypothesis of Fubini's theorem are satisfied: namely the function f is continous and the region of integration R is a "simple region" (between the graphs of two integrable functions), and thus Jordan measurable.

7. (10 points) Assume that $S \subset \mathbb{R}^2$ is a Jordan measurable set, that f, g are bounded functions defined on S, and that there exist sets $D_f \subset S$ and $D_g \subset S$ such that the Jordan measure $m(D_f) = m(D_g) = 0$ and

$$f \text{ is continuous at every point } x \in S \setminus D_f \qquad \& \qquad g \text{ is continuous at every point } x \in S \setminus D_g.$$

Does the product fg have to be integrable on S? Prove it or find a counterexample.

Solution.

Yes fg is integrable. This follows from 2 points:

- the discontinuity set of fg is contained in $D_f \cup D_g$; in other words, fg is continuous at every point where both f and g are continuous.
- The union of two zero measure sets also has zero measure; hence $m(D_f \cup D_g) = 0$.

Together these imply that the discontinuity set of fg is contained in a measure zero set, and it follows that fg is integrable.

THIS PAGE IS EMPTY. USE IT FOR SCRAP WORK. DO NOT TEAR OUT THIS PAGE.

What you write on this page WILL NOT BE MARKED

THIS PAGE IS EMPTY. USE IT FOR SCRAP WORK. DO NOT TEAR OUT THIS PAGE.

What you write on this page WILL NOT BE MARKED