(1) For each of the following sets, describe (without proof) the interior and boundary of the set, and circle whether or not the set is open, closed, or neither.

(a) $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \neq 1\} \subseteq \mathbb{R}^2$.
Solution. **Open**
Interior: S
Boundary: $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$

(b) $S = \mathbb{Q} \bigcup \{\pi\} \subseteq \mathbb{R}$
Solution. **Neither**
Interior: $\emptyset$
Boundary: $\mathbb{R}$

(c) The set $S = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1\} \subseteq \mathbb{R}^2$.
Solution. **Open**
Interior: S
Boundary: $\{(0, y) : y \in \mathbb{R}\} \cup \{(1, y) : y \in \mathbb{R}\}$

(2) (a) Prove that if $f : X \to Y$ is a function, and $B \subset Y$, then:

$$f[f^{-1}(B)] \subset B$$

Solution. Let $y \in f[f^{-1}(B)]$. Then there exists $x_0 \in f^{-1}(B)$ such that $y = f(x_0)$. By definition $f^{-1}(B)$ is the set of all $x \in X$ such that $f(x) \in B$. Therefore $y = f(x_0) \in B$ and hence $f[f^{-1}(B)] \subset B$.

(b) Prove that if $f : X \to Y$ is a surjective function, and $B \subset Y$, then:

$$f[f^{-1}(B)] \supset B$$

Solution. Let $b \in B$. Since f is surjective, there exists $x \in X$ such that $f(x) = b$. In particular, $x \in f^{-1}(B)$ since $f(x) = b \in B$. Thus $b = f(x) \in f[f^{-1}(B)]$ and hence $f[f^{-1}(B)] \supset B$.

(3) Let $S_1, \ldots, S_n$ be a finite collection of open sets. Prove that $\bigcap_{i=1}^n S_i$ is an open set.

Solution. If $x \in \bigcap_{i=1}^n S_i = \emptyset$ then it is open so suppose otherwise. Let $x \in \bigcap_{i=1}^n S_i$, so in particular $x \in S_i$ for each i. It suffices to show there is an $r > 0$ such that $B_r(x) \subseteq \bigcap_{i=1}^n S_i$. Since each S_i is open, there exist $r_i > 0$ such that $x \in B_{r_i}(x) \subseteq S_i$. Let $r = \min\{r_i : 1 \leq i \leq n\}$. Note that $r > 0$ since it is the minimum of a finite set of positive real numbers. Then $x \in B_r(x) \subseteq S_i$ for each i, hence $B_r(x) \subseteq \bigcap_{i=1}^n S_i$.

(4) Consider the set $A = \{\frac{1}{n} + \frac{1}{2^m} : m, n \in \mathbb{N}\}$. Is it a closed subset of $\mathbb{R}$? Justify your answer with a proof.

Solution. It is not a closed set. For example, the sequence $x_n = \frac{1}{n} + \frac{1}{2^n}$ lies in A, but its limit, which is 0, does not belong to A. Recall that a set is closed if and only if for every convergent sequence from the set, its limit as well lies in the set.

(5) Give an example of a sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}$ such that $\lim_{n\to\infty} |x_{n+1} - x_n| = 0$ but $(x_n)_{n=1}^{\infty}$ has no finite limit.

Solution. Take $x_n = \log n$. Then $\lim |x_{n+1} - x_n| = \lim \log(1 + \frac{1}{n}) = 0$, but x_n diverges (converges to ∞).

(6) Let $0 < a_0 < 1$ be a real number, and define the sequence $(a_n)_{n=1}^{\infty}$ by the recursive formula $a_{n+1} = a_n - a_n^2$. Prove that a_n converges and compute its limit.

Solution. Since $a_{n+1} = a_n - a_n^2 \leq a_n$, we see a_n is a non-increasing sequence. It is also easy to check by induction on n that $0 < a_n < 1$: indeed $0 < a_0 < 1$, and if $0 < a_n < 1$ then $a_{n+1} = a_n(1 - a_n)$ also satisfies $0 < a_{n+1} < 1$. It follows that a_n is a convergent sequence as it is non-increasing and bounded from below. Denoting $L = \lim a_n$, we can write

$$L = \lim a_n = \lim a_{n+1} = \lim(a_n - a_n^2) = L - L^2$$

where the last equality is obtained by arithmetics of limits. It follows that $L = L - L^2 \Rightarrow L^2 = 0 \Rightarrow L = 0$.

(7) A sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}^m$ with metric $d(,)$ is said to be a *fast Cauchy Sequence* if the series $\sum_{n=1}^{\infty} d(x_n, x_{n+1})$ converges. (Let's say the *series* converges to some number $L \in \mathbb{R}$).

(a) State the definition of a sequence $(x_n)_{n=1}^{\infty}$ being a Cauchy sequence.
Solution. The sequence $(x_n)_{n=1}^{\infty}$ is Cauchy if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $m, k > N$ then

$$d(x_m, x_k) < \epsilon.$$

(b) Prove that a *fast Cauchy sequence* is indeed a Cauchy sequence.
Solution. Let $\epsilon > 0$. Suppose $(x_n)_{n=1}^{\infty}$ is a fast Cauchy sequence. Since $\sum_{n=1}^{\infty} d(x_n, x_{n+1})$ converges, it is Cauchy and thus there exists $N \in \mathbb{N}$ such that if $m, k > N$ then

$$\left| \sum_{n=1}^{m} d(x_n, x_{n+1}) - \sum_{n=1}^{k} d(x_n, x_{n+1}) \right| < \epsilon.$$

Without loss of generality suppose $m > k$. By the triangle inequality

$$\begin{aligned} d(x_k, x_m) &\leq d(x_k, x_{k+1}) + d(x_{k+1}, x_{k+2}) + \cdots + d(x_{m-1}, x_m) \\ &= \sum_{n=1}^{m-1} d(x_n, x_{n+1}) - \sum_{n=1}^{k-1} d(x_n, x_{n+1}) \\ &= \left| \sum_{n=1}^{m-1} d(x_n, x_{n+1}) - \sum_{n=1}^{k-1} d(x_n, x_{n+1}) \right| \\ &< \epsilon. \end{aligned}$$

Therefore $(x_n)_{n=1}^{\infty}$ is Cauchy.