Mini-Problems 8

1. Clairaut's basic theorem says that if $f:\mathbb{R}^n\to\mathbb{R}$ is a C^2 function, then $\partial_{ij}f=\partial_{ji}f$ for all $1\le i,j\le n$. Use the basic theorem to prove the more general version of Clairaut's theorem, which is Theorem 2.40 of the class notes: if $f:\mathbb{R}^n\to\mathbb{R}$ is a C^k function, then $\partial_{i_1\cdots i_k}f=\partial_{j_1\cdots j_k}f$ whenever $(i_1,\ldots,i_k)$ and $(j_1,\ldots,j_k)$ are tuples of indices which are rearrangements of each other. It may be helpful to observe that any rearrangement of $(i_1,\ldots,i_k)$ may be obtained by a sequence of transpositions which switch i_r and i_{r+1} for some $1\le r\le k-1$.

2. Consider the function $f:\mathbb{R}^3\to\mathbb{R}$ given by $f(x,y,z)=e^{x+y}\sin(y+z)$. This function is (obviously) C^∞. (i) Calculate $\partial_{133}f$, $\partial_{313}f$, $\partial_{113}f$ and $\partial_{311}f$. If your calculation is correct then these won't all be equal. Why doesn't this contradict Clauraut's theorem? (ii) Calculate the Hessian of $g(x,y)=xe^{x+y}+y$.

3. Consider the following matrices. For each of them, figure out if they are equal to the Hessian of some C^∞ function $f:\mathbb{R}^2\to\mathbb{R}$, and if so, find all such functions. (i) $\begin{pmatrix}1 & 3x+y\\ 3y+x & 3x\end{pmatrix}$ (ii) $\begin{pmatrix}x-2y & x+2y\\ x+2y & 2x+2y\end{pmatrix}$ (iii) $\begin{pmatrix}2y+2 & 2x+4y\\ 2x+4y & 4x+4\end{pmatrix}$.

4. Suppose you have a C^∞ function $f:\mathbb{R}^n\to\mathbb{R}$ such that $\partial_{i_1\cdots i_k}f=\partial_{j_1\cdots j_k}f$ for **any** sets of indices $\{i_1,\ldots,i_k\}$ and $\{j_1,\ldots,j_k\}$ and any $k\ge 1$. What can you say about f? Can you find a characterization of all such functions?