**Mini-Problems 6**

**1.** Let $f : \mathbb{R} \to \mathbb{R}^n$ be a function such that $||f(t)|| = 1$ for all $t \in \mathbb{R}$. Prove that $f'(t) \cdot f(t) = 0$.

**2.** Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

Show that its partial derivatives exist everywhere and are bounded functions on all of $\mathbb{R}^2$ (this implies that $f$ is continuous). Nevertheless, show that $f$ is not differentiable at $(0,0)$. Hint: a sometimes useful way to prove that a function is not differentiable is to show that for some unit vector $u$ and point $a$, the directional derivative $\partial_u f(a) \neq \nabla f(a) \cdot u$ (see Theorem 2.17 of the notes).

**3.** Let $f : S \to \mathbb{R}^m$ be a differentiable function, where $S \subseteq \mathbb{R}^n$ is *connected* open set. Suppose that the Jacobian matrix $Df(x) = 0$ for every $x \in S$. Prove that $f$ is constant. What goes wrong if $S$ is not connected (the condition about openness is there just so that the derivative makes sense in $S$)?

**4.** Prove the following identities for $f, g : \mathbb{R}^n \to \mathbb{R}$, which are clearly generalizations of the 1-variable product and quotient rules for derivatives: (i) $\nabla(fg) = f\nabla g + g\nabla f$ and (ii) $\nabla(1/f) = -f^{-2}\nabla f$ wherever $f \neq 0$.