Mini-Problems 4

1. Determine whether the following limits exists or not: (i) $\lim_{(x,y)\to(0,0)} \frac{y^2(1-\cos(x))}{x^4+y^2}$ and (ii) $\lim_{(x,y)\to(0,0)} \frac{y^2+(1-\cos(x))^2}{x^4+y^2}$.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined everywhere except possibly the origin. If the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ exists and has the same value along all paths of the form $y = \alpha x^n$ for any $\alpha \in \mathbb{R}$, $n = 1, 2, \ldots$, then must the limit exist? Either prove this or find a counterexample.

3. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function. We define the *graph* of f to be the subset $\Gamma_f = \{(x, f(x)) : x \in \mathbb{R}^n\}$ of $\mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$. Prove that if f is continuous then Γ_f is a closed subset of $\mathbb{R}^{n+m}$. Also give a counterexample to the converse statement.

4. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function. Prove that $f(\bar{A}) \subseteq \overline{f(A)}$ for every subset $A \subseteq \mathbb{R}^n$. Give an example where the inclusion is strict.