Mini Problems 21

1. Let S be the surface defined by the equations $z = 14 - 8x^2 - 3y^2$ and $z \geq 0$, oriented with normal vector pointing in the positive z-direction. Let

$$F(x, y, z) = (z^2 e^{\sin(\cos(x))}, 2^z e^{e^{y^2}}, xyz).$$

Calculate

$$\int\int_S \nabla \times F \cdot dA.$$

(Hint: use Stokes' theorem to replace the integral over S with the integral over a different *surface*).

2. Let S be the surface defined by $z = e^{4-x^2-y^2}$ and $z \geq 1$, oriented with the normal vector pointing in the positive z-direction. Let $F(x, y, z) = (x^2, y, 1 - 2xz - z)$. Use the divergence theorem to calculate

$$\int\int_S F \cdot dA.$$

3. If f and g are smooth functions $\mathbb{R}^3 \to \mathbb{R}$, and S is a piecewise smooth orientable surface, show that

$$\int_{\partial S} (f \nabla g) \cdot ds = \int\int_S (\nabla f \times \nabla g) \cdot dA.$$

4. What is wrong with the following argument which purports to show that every vector field F in $\mathbb{R}^3$ is conservative (which you know to be false)?

Let S be a surface enclosing a region in which F is smooth. Since $\nabla \cdot (\nabla \times F)$ is identically zero we have by the divergence theorem that

$$\int\int_S \nabla \times F \cdot dA = \int\int\int_D \nabla \cdot (\nabla \times F)\, dV = 0$$

where D is the interior of S. On the other hand, by Stokes' theorem,

$$\int\int_S \nabla \times F \cdot dA = \int_{\partial S} F \cdot ds$$

so this line integral is always 0. Now use the fact that being conservative is equivalent to line integrals on simple closed curves vanishing.