Mini Problems 14

1. Using the definition of the Riemann integral, prove that if $f \geq 0$ is a continuous function on $[a, b]$ and $\int_a^b f(x)dx = 0$ then f is identically zero.

2. Is the function $f(x) : [0, 1] \to \mathbb{R}$ defined to be 2 if x has a decimal expansion containing the digit 2 infinitely many times and 0 otherwise integrable? Prove it.

3. Prove the subnormality of integrals: if f is integrable then so is $|f|$ and

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx.$$

Find an example of a function such that $|f|$ is integrable but f isn't.

4. Define $f(x) : [0, 1] \to \mathbb{R}$ to be 0 if x is irrational and $1/q$ if $x = p/q$ is rational in lowest terms. Prove that f is integrable, and that its integral is 0. Hint: It may be helpful to observe that each set $A_n = \{x : f(x) \geq 1/n\}$ is finite.