Mini Problems 12

1. Define $p : (0,1)\times(1,2)\times(-1,1)\to\mathbb{R}^5$ by $p(u,v,w) = (u+v^2, 3v, vw^2 + uvw, v-w, 2w)$. Show that the image of p is a smooth submanifold of $\mathbb{R}^5$. What is its dimension?

2. **(i)** The unit circle in $\mathbb{R}^2$ is obviously a smooth space, yet it has a parametrization that self-intersects given by $p:\mathbb{R}\to\mathbb{R}^2$, $p(t) = (\cos(t), \sin(t))$. Explain this. **(ii)** Does the union of two smooth submanifolds of $\mathbb{R}^n$ need to be a smooth submanfold? How about the intersection? What if we require that both of the submanifolds being unioned/intersected have the same dimension?

3. Determine whether the following spaces are smooth, and if so, determine their dimension: **(i)** the image of $p(t) = (\cos(t)\sin(t), \cos(t) + 2\sin(t))$ for $t \in (0, \pi/4)$ **(ii)** the zero-locus of $F : \mathbb{R}^4 \to \mathbb{R}^2$ given by $F(x,y,z,u) = (xy + 3ux + y^2 - 1, xy + uz)$ **(iii)** the image of $p(s,t) = (s\cos^2(t), s\sin^2(t), s^2\sin^2(2t)/2, s^2)$ **(iv)** the image of $p(t) = (\cos(|t|), \sin(|t|)$ for $t \in (-2\pi, 2\pi)$.

4. Consider the following setup: you have a 1-dimensional rod of length 2, one end of which is fixed to the origin in $\mathbb{R}^2$, and the other end of which is free to rotate around the origin. The free end of the rod is attached to a second rod of length 1, which is free to rotate around the end of the first rod (draw a picture). The set of possible physical positions (or "states") of this system of two rods can be described by the coordinates (x_1, y_1) and (x_2, y_2) of the free end of the first and second rod respectively.

Show that the set of possible states $\{(x_1, y_1, x_2, y_2)\} \subseteq \mathbb{R}^4$ is a smooth submanifold.