Mini Problems 11

1. Consider the function $F : \mathbb{R}^2 \to \mathbb{R}$ given by $F(x, y) = xy^2 + 1$. Use the implicit function theorem to determine which of the level sets of F are locally graphs of C^1 functions of a single variable. Prove your answer (that is, prove that the left-over level sets are *not* locally graphs of single variable functions). [Recall that the level sets of F are the sets $\{(x, y) \in \mathbb{R}^2 : F(x, y) = c\}$ for $c \in \mathbb{R}$].

2. Recall the cylindrical coordinate system in $\mathbb{R}^3$:

$$\begin{cases} x = & r\cos\theta \\ y = & r\sin\theta \\ z = & z. \end{cases}$$

Near which points in $\mathbb{R}^3$ can we solve for r, θ and z in terms of the Cartesian coordinates x, y, z? As well as a proof, give a geometric reason for your answer.

3. Consider the curve C defined by $F(x, y) = 0$ in $\mathbb{R}^2$ where $F(x, y) = x^2 - y^3$. Observe that $(0, 0)$ lies on C and that $F_y(0, 0) = 0$. Yet show that it is possible to describe C as the graph of a function $y = f(x)$. Does this contradict the implicit function theorem?

4. Consider the system of equations

$$\begin{cases} x_1y_2^2 - 3x_2y_3 & = -1 \\ x_1y_1^5 + 3x_2y_2 - 4y_2y_3 & = 13 \\ x_2y_1 + 3x_1y_3^2 & = -27. \end{cases}$$

(i) Show that near the point $(-1, 0, -1, -1, 3)$ it is possible to solve for y_1, y_2, y_3 as functions of x_1 and x_2. (ii) Find $\frac{\partial y_1}{\partial x_1}(-1, 0)$, $\frac{\partial y_2}{\partial x_1}(-1, 0)$ and $\frac{\partial y_3}{\partial x_1}(-1, 0)$.