University of Toronto

Faculty of Arts and Science
Math 237Y1Y-Advanced Calculus
Term Test 3
Friday, January 29, 2016
Duration - 110 minutes
No Aids Permitted

Last name: ____________________

First Name: ____________________

Student Number: ____________________

The "PENCIL=NO REMARKING" POLICY. If you choose to write your test in pencil, you forefeit your opportunity for a remark. Therefore, please WRITE YOUR TEST IN PEN.

Tutorial:

T0201	T0401	T0601	T0701	T5100	T5101	T5102	T5201	T5301
M4 RS208	T4 SS2108	W4 BA1230	R4 SS2108	M5 RS 208	T5 SS2108	M5 SS1070	W5 GB304	R5 SS2108
Travis	Anne	Ren	Travis	Travis	Anne	BEN	Ren	BEN

This exam contains 9 pages (including this cover page) and 7 problems. Check to see if any pages are missing and ensure that all required information at the top of this page has been filled in.

No aids are permitted on this examination. Examples of aids that are NOT permitted include, but are not limited to textbooks, notes, calculators, or any electronic device.

Unless otherwise indicated, you are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total:	70	

1. (10 points) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = x^2y + xy^2 - xy$. Find the critical points of f and characterize them as local minimal, local maximal, saddle, or degenerate critical points.

2. (10 points) Find the extremal values of $f(x, y, z) = x^2 - y^2$ on the surface defined by $x^2 + y^2 + z^2 = 1$, and find all points where the extrema are achieved.

3. (10 points) Find the 3rd order Taylor polynomials at $(0,0)$ for the functions $f(x,y) = \sin(x+y)$ and $g(x,y) = \sin(x)\cos(y)$.

4. (10 points) Show that the following system always has a solution for sufficiently small a,

$$\begin{aligned} x + y + \sin(xy) &= a \\ \sin(x^2 + y) &= 2a \end{aligned}$$

5. (10 points) Let U denote the open subset $\{(a,b) \in \mathbb{R}^2 | a^2 + b^2 < 1\} \subset \mathbb{R}^2$. Suppose that $f : U \to \mathbb{R}$ is a differentiable function and that $Df(a,b) = 0$ for all points $(a,b) \in U$. Show that f is constant.
(Hint: use the Mean Value Theorem.)

6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 differentiable function. Let

$$\begin{aligned} G(x,y,u) &= f(x,y) + u^2 \\ H(x,y,u) &= ux + 3y^3 + u^3 \end{aligned}$$

Let the constraints $G(x,y,u) = 0$ and $H(x,y,u) = 0$ have a solution $(x_0, y_0, u_0) = (2, 1, -1)$.

(a) (6 points) Find the conditions on $f_x(x_0, y_0)$, $f_y(x_0, y_0)$ (if any) that ensure that there exist C^1 differentiable functions $x = g(y)$ and $u = h(y)$ defined on an open set in $\mathbb{R}$ that satisfy both equations, and such that $g(y_0) = x_0$ and $h(y_0) = u_0$.

(b) (4 points) Using part (a), and assuming that $Df(x_0, y_0) = \begin{bmatrix} 1 & -3 \end{bmatrix}$, write a system of two equations that will allow you to solve for the two unknowns $g'(y_0)$, and $h'(y_0)$. (Note: you are not being asked to actually solve the system.)

7. (10 points) Let U denote the open unit disk $\{(a,b) \in \mathbb{R}^2 | a^2 + b^2 < 1\} \subset \mathbb{R}^2$. Suppose that $f : U \to \mathbb{R}$ is a C^4-function and that

   - $\partial^\alpha(f)(0,0) = 0$ for all α with $0 \leq |\alpha| \leq 2$;
   - $\partial^{(3,0)}(f)(0,0) = 1$; and
   - $\partial^\alpha(f)(a,b) \leq 1$ for all $(a,b) \in U$ and α with $|\alpha| = 4$.

   Show that $f(a,0) > 0$ for $0 < a < 1$ and $f(a,0) < 0$ for $-1 < a < 0$.

THIS PAGE IS EMPTY. USE IT FOR SCRAP WORK.

What you write on this page will not be marked, unless you write next to the relevant question 'continued on last page'