Test 3 Solutions

Problem 1 a) $f(x, y) = e^{x+y}$. We have

α	$\partial^\alpha f(0,0)$	$\alpha!$
$(0,0)$	1	1
$(1,0)$	1	1
$(0,1)$	1	1
$(2,0)$	1	2
$(1,1)$	1	1
$(2,0)$	1	2
$(3,0)$	1	6
$(2,1)$	1	2
$(1,2)$	1	2
$(0,3)$	1	6

$$f(x, y) = 1 + x + y + \frac{1}{2}x^2 + \frac{1}{2}y^2 + xy + \frac{1}{6}x^3 + \frac{1}{2}x^2y + \frac{1}{2}xy^2 + \frac{1}{6}y^3 + R_f(x, y)$$

b) $g(x, y) = \sin(xy)$. Since $\sin(xy) = xy + \frac{1}{6}(xy)^3 + \cdots$ then up to order 3, we have

$$g(x, y) = xy + R_g(x, y)$$

Problem 2 $f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$

a) $\nabla f = (-6x+6y, 6y-6y^2+6x) = 0$ and thus $p_0 = (0, 0)$, $p_1 = (2, 2)$ are two critical points. The function is differentiable everywhere and there is no other critical points.

b) At p_0, the Hessian matrix is

$$H_f(0, 0) = \begin{pmatrix} -6 & 6 \\ 6 & 6 \end{pmatrix},$$

and therefor p_0 is a saddle point. At p_1, the Hessian matrix is

$$H_f(2, 2) = \begin{pmatrix} -6 & 6 \\ 6 & -18 \end{pmatrix},$$

and thus p_1 is a maximum.

Problem 3 Consider two curves $y = x^2$ and $q = p - 1$. Two solutions are given.

i. Minimize the conditional minimization problem

$$\min(x - p)^2 + (x^2 - q)^2 \text{subject to} q = p - 1$$

Make the function

$$F(x, p, q, \lambda) = (x - p)^2 + (x^2 - q)^2 - \lambda(q - p + 1).$$

For $\nabla F = 0$, we obtain

$$\begin{cases} 2(x-p) + 4x(x^2-q) = 0 \\ -2(x-p) + \lambda = 0 \\ -2(x^2-q) - \lambda = 0 \\ q = p - 1 \end{cases}.$$

Solving above system, gives $x = \frac{1}{2}$, $p = \frac{7}{8}$, $q = \frac{-1}{8}$ and $\lambda = \frac{-3}{4}$. Therefore, $R = \frac{3}{4\sqrt{2}}$.

ii. Form the minimization problem

$$F(x,y,p,q,\lambda) = (x-p)^2 + (y-q)^2 - \lambda_1(y-x^2) - \lambda_2(q-p+1)$$

Solving the equation gives again $x = \frac{1}{2}, y = \frac{1}{4}$, $p = \frac{7}{8}$ and $q = \frac{-1}{8}$, $\lambda_1 = 1$, $\lambda_2 = -1$ and $R = \frac{3}{4\sqrt{2}}$.

Problem 4 $f(\theta,\varphi) = (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)$.

a) $f(t,\pi/2) = (\cos t, \sin t, 0) \Rightarrow S_1 : x^2+y^2 = 1$. $f(0,t) = (\sin t, 0, \cos t) \Rightarrow S_2 : x^2+z^2 = 1$

b) $\frac{\partial f}{\partial \theta}(0,\pi/2) = (0,1,0)$, $\frac{\partial f}{\partial \varphi}(0,\pi/2) = (0,0,-1)$.

c) These are the same as $\frac{\partial f}{\partial \theta}(0,\pi/2)$ and $\frac{\partial f}{\partial \varphi}(0,\pi/2)$.

Problem 5 a) Define $f(x,y) = \ln y + xy - 1$. It is simply verified that $f(1,1) = 0$. The function f is C^1 in a neighborhood of $(1,1)$ and furthermore $\partial_y f(1,1) = 2 \neq 0$ and thus there is an interval $I = (1-\delta, 1+\delta)$ and a C^1 function $y : I \to \mathbb{R}$ such that $f(x,y(x)) = 0$.

b) $F(\vec{x},y) = (\vec{x}, f(\vec{x},y))$ where $\vec{x} \in \mathbb{R}^n$ and $f : \mathbb{R}^{n+1} \to \mathbb{R}$. We have

$$DF(\vec{x}_0,y_0) = \begin{pmatrix} I_{n\times n} & 0 \\ \nabla^t_{\vec{x}} f & \frac{\partial f}{\partial y}(\vec{x}_0,y_0) \end{pmatrix}.$$

We observe that $\det DF(\overrightarrow{x_0},y_0) = \frac{\partial f}{\partial y}(\vec{x}_0,y_0)$ and therefore DF is invertible at $(\vec{x}_0,y_0)$ iff $\frac{\partial f}{\partial y}(\vec{x}_0,y_0) \neq 0$.

Problem 6 Define $f_1(x,y) = x + y + \sin(xy)$, $f_2(x,y) = \sin(x^2+y)$ and the C^1 function $F(x,y) = (f_1(x,y), f_2(x,y))$. We have $F(0,0) = (0,0)$ and furthermore $DF(0,0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is invertible. The inverse function theorem guarantees the existence of open sets U,V of $(0,0)$ such that $F : U \to V$ is one to one and onto. Choose a small enough such that $(a,2a) \in V$ and thus $F^{-1}(a,2a) \in U$.

Problem 7 Let $\varepsilon = 1$ and assume $\delta > 0$ exists such that if $|x-y| < \delta$ then $|x^2-y^2| < 1$ for all $x,y \in [0,\infty)$. Choose $y = x + \frac{\delta}{2}$ and thus

$$\frac{\delta}{2}\left(2x + \frac{\delta}{2}\right) < 1,$$

for all $x \in [0,\infty)$. If we choose $x = \frac{1}{\delta}$, we obtain $1 + \frac{\delta^2}{4} < 1$ that is impossible.