Math 237Y- 2016-2017

Term Test 2 - November 18, 2016

Time allotted: 110 minutes. Aids permitted: None.

Total marks: 70

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Last First

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Instructions

- DO NOT WRITE ON THE QR CODE at the top of the pages.
- DO NOT DETACH ANY PAGE.
- NO CALCULATORS or other aids allowed.
- Unless otherwise stated, you must JUSTIFY your work to receive credit.
- Check to make sure your test has all 10 pages.
- You can use the last two pages as scrap paper.
- DO NOT START the test until instructed to do so.

GOOD LUCK!

1. Let A and B be two connected subsets of $\mathbb{R}^2$

(a) (5 points) Must $A \cup B$ be connected? Provide a proof or a counterexample.

(b) (5 points) Must $A \cap B$ be connected? Provide a proof or a counterexample.

solution

a. This is false. For example, let $A = \{p\}$ and $B = \{q\}$, where p and q are two distinct points. Then $S_1 = A, S_2 = B$ yields a disconnection of $A \cup B = \{p, q\}$.

b. This is false. For example, let

$$A := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1, x \geq 0\}, \qquad B := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1, x \leq 0\}.$$

Then

$$A \cap B = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1, x = 0\}.$$

This set consists of exactly two points, $(0,1)$ and $(0,-1)$. So taking $S_1 = \{(0,1)\}$ and $S_2 = \{0,-1)\}$ yields a disconnection of $A \cap B$.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ and $g : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the equations

$$\begin{aligned} f(x,y) &= (e^{2x+y}, 3y - \cos x, x^2 + y + 2) \\ g(u,v,w) &= (3u + 2v + w^2, u^2 - w + 1) \end{aligned}$$

Let $h(x,y) = g(f(x,y))$

(a) (7 points) Find $Dh(x,y)$.

(b) (3 points) Find $Dh(0,0)$.

solution

a. We have $Df(x,y) = \begin{pmatrix} 2e^{2x+y} & e^{2x+y} \\ \sin x & 3 \\ 2x & 1 \end{pmatrix}$ and $Dg(u,v,w) = \begin{pmatrix} 3 & 2 & 2w \\ 2u & 0 & -1 \end{pmatrix}$ so by the chain rule

$$\begin{aligned} Dh(x,y) &= (Dg(f(x,y)))(Df(x,y)) \\ &= \begin{pmatrix} 3 & 2 & 2w \\ 2u & 0 & -1 \end{pmatrix} \begin{pmatrix} 2e^{2x+y} & e^{2x+y} \\ \sin x & 3 \\ 2x & 1 \end{pmatrix} \\ &= \begin{pmatrix} 6e^{2x+y} + 2\sin x + 4x(x^2+y+2) & 3e^{2x+y} + 6 + 2(x^2+y+2) \\ 4e^{2(2x+y)} - 2x & 2e^{2(2x+y)} - 1 \end{pmatrix} \end{aligned}$$

b. Plugging in $(x,y) = (0,0)$ we get $Dh(0,0) = \begin{pmatrix} 6 & 13 \\ 4 & 1 \end{pmatrix}$

3. (10 points) Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ be C^1 functions and define $F(x, y) = \displaystyle\int_{f(x,y)}^{g(x,y)} h(t)\mathrm{d}t$.

Compute $\dfrac{\partial F}{\partial x}$ and $\dfrac{\partial F}{\partial y}$.

solution

Consider the function $I(s) = \displaystyle\int_0^s h(t)dt$. By the fundamental theorem of calculus $I'(s) = h(s)$. Then $F(x, y) = I(g(x, y)) - I(f(x, y))$ so by the chain rule

$$\frac{\partial}{\partial x}F(x, y) = I'(g(x, y))\frac{\partial g}{\partial x} - I'(f(x, y))\frac{\partial f}{\partial x} = h(g(x, y))\frac{\partial g}{\partial x} - h(f(x, y))\frac{\partial f}{\partial x}$$

and similarly

$$\frac{\partial}{\partial y}F(x, y) = I'(g(x, y))\frac{\partial g}{\partial y} - I'(f(x, y))\frac{\partial f}{\partial y} = h(g(x, y))\frac{\partial g}{\partial y} - h(f(x, y))\frac{\partial f}{\partial y}.$$

4. (10 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 5 + \dfrac{5x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 5 & \text{if } (x,y) = (0,0) \end{cases}$$

Prove that f is continuous at $(0,0)$ by using only the ϵ-δ definition.

solution

Let $\epsilon > 0$. Then

$$|f(x,y) - f(0,0)| = \left| \frac{5x^2y}{x^2+y^2} \right| = \frac{5x^2|y|}{x^2+y^2} \leqslant \frac{5(x^2+y^2)|y|}{x^2+y^2} = 5|y|.$$

So if we let $\delta = \epsilon/5$ then for all $(x,y) \in B_\delta((0,0))$,

$$|f(x,y) - f(0,0)| \leqslant 5|y| \leqslant 5\sqrt{x^2+y^2} < 5\delta = 5\frac{\epsilon}{5} = \epsilon.$$

Thus f is continuous at $(0,0)$.

5. (10 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Prove that f is not differentiable at $(0,0)$.

solution

Notice that $f = 0$ whenever $x = 0$ or $y = 0$. Let us use the **limit definition of the partial derivative** to find that $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$.

(**Note**: a common mistake was to partially differentiate the function $\dfrac{xy^2}{x^2+y^2}$ instead of the "two-pieces" function f defined above.)

Let $\mathbf{a} = (0,0)$, $\mathbf{e_1} = (1,0)$, $\mathbf{e_2} = (0,1)$.

$\frac{\partial f}{\partial x}(\mathbf{a}) = \lim_{t\to 0} \dfrac{f(\mathbf{a}+t\mathbf{e_1}) - f(\mathbf{a})}{t} = \lim_{t\to 0} \dfrac{f(t\mathbf{e_1})}{t} = \lim_{t\to 0} \dfrac{\frac{t(0^2)}{t^2+0^2}}{t} = \lim_{t\to 0} \dfrac{0}{t} = 0$

$\frac{\partial f}{\partial y}(\mathbf{a}) = \lim_{t\to 0} \dfrac{f(\mathbf{a}+t\mathbf{e_2}) - f(\mathbf{a})}{t} = \lim_{t\to 0} \dfrac{f(t\mathbf{e_2})}{t} = \lim_{t\to 0} \dfrac{\frac{0(t^2)}{0^2+t^2}}{t} = \lim_{t\to 0} \dfrac{0}{t} = 0.$

Now, let $\mathbf{u} = (1,1)$, and let's compute $\partial_{\mathbf{u}} f(\mathbf{a})$.

$\partial_{\mathbf{u}} f(\mathbf{a}) = \lim_{t\to 0} \dfrac{f(\mathbf{a}+t\mathbf{u}) - f(\mathbf{a})}{t} = \lim_{t\to 0} \dfrac{f(t\mathbf{u})}{t} = \lim_{t\to 0} \dfrac{\frac{t(t^2)}{t^2+t^2}}{t} = \lim_{t\to 0} \dfrac{t^3}{2(t^3)} = \frac{1}{2}.$

This will lead to a contradiction, if we assume that f is differentiable at $\mathbf{a} = (0,0)$. For in that case, we should have that

$$\partial_{\mathbf{u}} f(0,0) = u \cdot \nabla f(0,0) = \mathbf{u} \cdot (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})(0,0) = 0.$$

Since $0 \neq \frac{1}{2}$, we conclude that f cannot be differentiable at $\mathbf{a} = (0,0)$.

6. (10 points) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real valued function such that there exists a $M > 0$ and $\alpha > 0$ such that for all x, y in $\mathbb{R}^n$, $|f(x) - f(y)| \leqslant M\|x - y\|^{1+\alpha}$.

(a) (7 points) Prove (by using only the definition as a limit) that f is differentiable at $a \in \mathbb{R}^n$ and that $\nabla f(a) = \mathbf{0}$.

(b) (3 points) Prove that f is a constant function.

solutions

a. It suffices to show that $\lim_{h\to 0} \dfrac{|f(a+h) - f(a)|}{\|h\|} = 0$. But

$$\frac{|f(a+h) - f(a)|}{\|h\|} \leqslant M\frac{\|h\|^{1+\alpha}}{\|h\|} = M\|h\|^{\alpha} \to 0$$

as $h \to 0$. Thus f is differentiable at a with $\nabla f(a) = 0$.

b. Let $x, y \in \mathbb{R}^n$ be arbitrary. By part (a) f is everywhere differentiable and since $\mathbb{R}^n$ contains the line segment between x and y we can use the mean value theorem which gives us for some c

$$f(x) - f(y) = \nabla f(c) \cdot (x - y) = 0 \cdot (x - y) = 0$$

so $f(x) = f(y)$. Since x, y were arbitrary it follows that f is constant.

7. (10 points) Let $S \subset \mathbb{R}$ be a compact set and let $f : \mathbb{R} \to \mathbb{R}$ be a continous function. Let us define $G := Graph_S(f) := \{(x, y) \in \mathbb{R}^2 : x \in S, y = f(x)\}$. Prove that G is a compact set.

solution Consider the function $g : \mathbb{R} \to \mathbb{R}^2$ defined by $g(x) = (x, f(x))$. We claim that g is continuous. Let $a \in \mathbb{R}$ and suppose (x_n) is a sequence converging to a. We must show that $g(x_n) = (x_n, f(x_n))$ converges to $g(a) = (a, f(a))$. But this sequence converges if and only if each component converges. By assumption, $x_n \to a$ and by continuity of f we have $f(x_n) \to f(a)$. Thus $g(x_n) \to g(a)$ and so g is continuous. Then $G = g(S)$ which is the image of a compact set under a continuous function and therefore is continous.

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