University of Toronto

Faculty of Arts and Science
Math 237Y1Y-Advanced Calculus
Term Test 4, Winter 2016
Duration - 110 minutes
No Aids Permitted

Last name: ______________________________

First Name: ______________________________

Student Number: ______________________________

The "PENCIL=NO REMARKING" POLICY. To ensure the integrity of the grades, we are reintroducing the following policy for the winter term: If you choose to write your test in pencil, you forefeit your opportunity for a remark. Therefore, it is strongly suggested that you WRITE YOUR TESTS IN PEN.

Tutorial:

T0201	T0401	T0601	T0701	T5100	T5101	T5102	T5201	T5301
M4 RS208	T4 SS2108	W4 BA1230	R4 SS2108	M5 RS 208	T5 SS2108	M5 SS1070	W5 GB304	R5 SS2108
Travis	Anne	Ren	Travis	Travis	Anne	BEN	Ren	BEN

This exam contains 10 pages (including this cover page) and 7 problems. Check to see if any pages are missing and ensure that all required information at the top of this page has been filled in.

No aids are permitted on this examination. Examples of aids that are NOT permitted include, but are not limited to textbooks, notes, calculators, or any electronic device.

Unless otherwise indicated, you are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total:	70	

1. Give a parameterization for each of the following sets. Be sure to indicate the domain of your parameterization.

   (a) (4 points) The ellipse $\{(x, y) \in \mathbb{R}^2 : a^2x^2 + b^2y^2 = c^2\}$ for $a, b, c > 0$.

   (b) (4 points) The elliptic paraboloid $\{(x, y, z) \in \mathbb{R}^3 : -x + 10y^2 + 2z^2 - 4 = 0\}$

   (c) (2 points) The solid cylinder $\{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 \leq 9, |y| \leq 1\}$.

2. (a) (2 points) State Fubini's Theorem.

(b) (2 points) Let $\alpha \in \mathbb{R}$ be an arbitrary non-zero constant. Compute

$$\int \frac{x-\alpha}{(x+\alpha)^3}\mathrm{d}x.$$

[Hint: To integrate $x/(x+\alpha)^3$ make the substitution $u = x + \alpha$]

(c) (4 points) Let R be the rectangle $R = [0,1] \times [0,1]$ and compute the iterated integrals

$$\iint_R \frac{x-y}{(x+y)^3}\,\mathrm{d}x\,\mathrm{d}y, \qquad \iint_R \frac{x-y}{(x+y)^3}\,\mathrm{d}y\,\mathrm{d}x.$$

[Notice that the order of integration is changed! Hint: You can use symmetry to calculate the second integral from the first integral.]

(d) (2 points) You should have found in part (b) that the integrals did not agree. Explain why this is not a contradiction to Fubini's theorem.

3. (10 points) Let D be the region in the first quadrant bounded by the the curves $x = 0$, $y = 2$ and $y = \sqrt[3]{x}$. Determine

$$\iint_D \frac{x}{y^7 + 2} \mathrm{d}A.$$

4. Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(u, v) = (x, y) = (u^2 - v^2, 2uv)$.

   (a) (3 points) Find the determinant of the jacobian matrix of T.

   (b) (7 points) Let $A = \big\{(u, v) \in \mathbb{R}^2 : 0 \le u \le 1, 0 \le v \le 1)\big\}$ be the unit square in $\mathbb{R}^2$. Find the image of A under the transformation T. Draw a picture of the image set and label the formulas of all the boundary curves.

5. Let $E : x^2 + y^2 + z^2 \leq 1$ be the closed unit ball in $\mathbb{R}^3$.

   (a) (4 points) Write out the parametrization of E in terms of the spherical coordinates (r, ϕ, θ) with ϕ representing the polar angle. Make sure to specify the domain of the parametrization.

   (b) (6 points) Evaluate $\iiint_E \exp{(x^2 + y^2 + z^2)^{3/2}} dV$. (Hint: The volume element is given by $dV = r^2 \sin\phi d\phi d\theta dr$.)

6. (10 points) Let $f, g : [a, b] \to \mathbb{R}$ be bounded integrable functions, with $\int_a^b f = I$ and $\int_a^b g = J$. Using any definition of integrability, show from first principles that

$$\int_a^b [2f - 3g] = 2I - 3J.$$

7. (a) (5 points) If $S = \{x_1, \ldots, x_n\}$ is a finite set of n vectors in $\mathbb{R}^2$, prove that S has content zero (Jordan measure zero) in $\mathbb{R}^2$.

(b) (5 points) Let $f : [a, b] \to \mathbb{R}$ be a bounded, (Riemann) integrable function. Prove that the graph of f, $\Gamma(f) := \{(x, f(x)) : x \in [a, b]\} \subseteq \mathbb{R}^2$ has content zero (Jordan measure zero).

THIS PAGE IS EMPTY. USE IT FOR SCRAP WORK.

What you write on this page will not be marked, unless you write next to the relevant question ‘continued on last page’